The higher-times approach to multisoliton solutions of the Harry Dym equation

1993 J. Phys. A: Math. Gen. 26 6005

(http://iopscience.iop.org/0305-4470/26/21/037)

J. Phys. A: Math. Gen. 26 (1993) 6005–6020. Printed in the UK

The higher-times approach to multisoliton solutions of the Harry Dym equation

L A Dmitrieva

Department of Mathematical and Computational Physics, Institute for Physics, St Petersburg University, St Petersburg 198904, Russia

Received 5 April 1993

Abstract. An N-cusp soliton solution of the Harry Dym equation (HDE) is constructed explicitly. The proposed approach is based on the algebraic 'effectivization' of the reciprocal links between the systems consisting of N higher-order analogues of the HDE and the corresponding analogues of the KdV equation. An important role in the discussed scheme belongs to the reciprocal auto-Bäcklund transformation for the Dym hierarchy.

1. Introduction

Multisoliton solutions of the non-linear evolution equations have been obtained by different methods. Among them one can mention the inverse scattering transform (IST), the Bäcklund transformation technique, the Hirota bilinear method, and the use of links with well studied equations. The present paper deals with the Harry Dym equation (HDE) $r_t = r^3 r_{xxx}$ [1] for which an explicit derivation of multisoliton solutions in the framework of the above methods is either impossible in principle or meets such analytical difficulties that make it practically impossible (see section 2). To all appearances this is the reason that the multisoliton solutions of the HDE (except the one-soliton case) have not yet been reported explicitly.

The HDE belongs to a special class of integrable non-linear evolution equations found by Wadati *et al* [2]. It has long been known that the HDE is related to the classical string problem [3–5], can be 'tractable' by the IST [6], has a bi-Hamiltonian structure [7], and possesses an infinite number of conservation laws [5, 7, 8] and symmetries [9, 10]. Also known are some properties of auto-Bäcklund transformations [16] for it.

Being related to the KdV equation via reciprocal links [13–19] and to the mKdV equation by the Ishimori transformation [11, 12], the HDE belongs to a class of non-linear evolution equations of 'not normal type' [20] and possesses some specific features which separate it from 'usual' integrable systems in $1 + 1$ dimensions. For example, a set of an infinite number of the HDE's higher-order analogues called the Dym hierarchy [5, 16] has the abundant symmetry structure [20]. The HDE also possesses the so-called 'weak-Painlevé' property [14, 15] rather than the usual Painlevé one.

Another essential point is that the one-soliton solution of the HDE (the so-called cusp soliton) which was initially obtained by the IST [6] and later derived by the 'direct integration method' [19] cannot be expressed in a closed form and has an implicit

0305-4470/93/216005+16$07.50

nature. It reads as follows:

$$r(x,t) = \tanh^2\left[\tfrac{1}{2}(p_1 x + p_1\varepsilon(x,t) + p_1^3 t - \zeta_1^0)\right] \tag{1}$$

where p_1, ζ_1^0 are real constants. The transendental phase $\varepsilon(x,t)$ is defined implicitly by the equation

$$\varepsilon(x,t) = \frac{2}{p_1}\tanh\left[\frac{1}{2}(p_1 x + p_1\varepsilon(x,t) + p_1^3 t - \zeta_1^0\right] + \frac{2}{p_1} \tag{2}$$

and is related to $r(x,t)$ by the equation

$$\varepsilon(x,t) = \int_{-\infty}^{x}\left(\frac{1}{r(x',t)} - 1\right)\mathrm{d}x' . \tag{3}$$

Notice that (1) and (2) describe the one-soliton solution completely and have the form which cannot be improved analytically. In this sense the one-soliton solution of the HDE implicit by its nature has been constructed explicitly.

As a natural generalization of (1) and (2) the N-soliton solution of the HDE can be expected to have the form

$$r(x,t) = F(\zeta_1,\ldots,\zeta_N) \qquad r\to 1 \quad \text{as} \quad |x|\to\infty \tag{4}$$

$$\zeta_k = p_k x + p_k\varepsilon(x,t) + p_k^3 t - \zeta_k^0 \qquad k = 1,\ldots,N \tag{5}$$

where p_k, ζ_k^0 are real constants, the phase $\varepsilon(x,t)$, being related to $r(x,t)$ by (3), is defined by

$$\varepsilon(x,t) = G(\zeta_1,\ldots,\zeta_N) \qquad \begin{cases} \varepsilon\to 0 & \text{as} \quad x\to -\infty \\ \varepsilon\to \text{constant} & \text{as} \quad x\to +\infty . \end{cases} \tag{6}$$

Equations (4)–(6) will describe the N-soliton solution of the HDE completely if one finds the functions $F(\zeta_1,\ldots\zeta_N)$ and $G(\zeta_1,\ldots,\zeta_N)$ explicitly. This is the main aim of the present paper.

Our approach is based on the joint integration of the system which consists of the first N HDE's higher-order analogues. In other words, we assume that the HDE N-soliton solution depends on $N-1$ 'higher times' t_m, $m = 2,\ldots,N$, and obeys the above system. Another essential point is an 'effectivization' of the links between the Nth-order HD system and the Nth-order KdV systems. By 'effectivization' we imply the procedure of deriving novel algebraic relations between the quantities related by simple transformations to the solutions of these systems. As a result, we construct the *implicit-by-its-nature N-soliton solution of the HDE explicitly. Since this solution* splits apart at the limit $|t|\to\infty$ into N cusp solitons, it is natural to call this solution as the N-cusp soliton solution.

Let us emphasize that the HDE was recently found to be relevant to the so-called Saffman–Taylor problem, i.e. the problem of the displacement of a viscous fluid by a less viscous one in a Hele–Shaw cell [21–23].

The organization of this paper is as follows. In section 2 we explain why the traditional use of the links between the HDE and the KdV or mKdV equations as well as an application of the IST give only the parametric representation for the HDE multisoliton solutions and therefore are not effective from the analytic point of view. In section 3 we present the scheme of constructing the N-cusp soliton solution of the HDE and its higher-order analogues and describe the final results.

2. Preliminaries

In this section we discuss briefly the known links between the HDE and the KdV equation [13–19] or the mKdV equation [11, 12]. The result of the IST application for the N-soliton case derived on the basis of [6] is also considered here. We show why the direct use of the above links and the IST constructions are not effective from analytic point of veiw for explicit derivation of the HDE multisoliton solutions.

The transformation from the KdV equation

$$u_\tau - 6uu_y + u_{yyy} = 0 \qquad u = u(y, \tau)$$

considered with the boundary conditions $u \to 0$ as $|y| \to \infty$, to the HDE

$$r_t = r^3 r_{xxx} \qquad r = r(x, t) \tag{7}$$

with the boundary conditions $r \to 1$ as $|x| \to \infty$, involves the following steps.

Firstly, one has to consider the Miura transformation

$$u = \tfrac{1}{4}(v^2 - 2v_y)$$

which gives the Riccati equation for $v(y, \tau)$. The function $v(y, \tau)$ is known to obey the mKdV equation

$$v_\tau - \tfrac{3}{2}v^2 v_y + v_{yyy} = 0 \,. \tag{8}$$

The further use of the Cole–Hopf transformation

$$v = \frac{R_y}{R}$$

leads to the function $R(y, \tau)$ with the boundary conditions $R \to 1$ as $|y| \to \infty$.

Finally, the reciprocal transformation

$$x = \int^y R(y', \tau)\,\mathrm{d}y' \qquad \tau = -t \tag{9}$$

gives the function

$$r(x, t) = R(y, \tau) \tag{10}$$

($r \to 1$ as $|x| \to \infty$), which obeys the HDE (7).

Remark. One can take the Miura transformation in the form

$$u = \tfrac{1}{4}(v'^2 + 2v'_y)$$

where $v'(y, \tau)$ also solves the mKdV equation (8). Then the function $R'(y, \tau)$ introduced by the Cole–Hopf transformation $v' = R'_y / R'$ defines the HDE solution $r'(x', t')$ which is related to $r(x, t)$ by the reciprocal auto-Bäcklund transformation [16].

Another form of the same transformation implies the start from the mKdV equation (8) written as

$$\Theta_\tau - \tfrac{1}{2}\Theta_y^3 + \Theta_{yyy} = 0 \tag{11}$$

where $\Theta_y = v$ and $\Theta \to 0$ as $|y| \to \infty$. The HDE solution $r(x,t)$ is then obtained by the 'Ishimori transformation' [12]

$$x = \int^y \exp\Theta \, \mathrm{d}y' \quad \tau = -t \qquad r(x,t) = R(y,\tau) = \exp\Theta(y,\tau)\,. \tag{12}$$

The analysis of both the above transformations shows that the obtained HDE solution is intrinsically implicit. Indeed, let

$$E(y,\tau) = \int_{-\infty}^{y} (1 - R(y',\tau))\,\mathrm{d}y'\,. \tag{13}$$

Then (9) and (10) can be written as

$$y = x + \varepsilon(x,t) \tag{14}$$

where

$$\varepsilon(x,t) = E(y,\tau) \tag{15}$$

and

$$r(x,t) = R(y,\tau)\,. \tag{16}$$

Knowing the functions $E(y,\tau)$ and $R(y,\tau)$, (14)–(16) describe the HD 'decreasing' ($r \to 1$ as $|x| \to \infty$) solution completely: the phase $\varepsilon(x,t)$ is defined implicitly by (15) and the solution itself is given explicitly by (16).

Notice that on use of (15), (13) and (9) the function $\varepsilon(x,t)$ can be written as in (3). Hence, the one-soliton solution of the HDE possesses exactly the form (14)–(16) with

$$R = \tanh^2\left[\frac{p}{2}(y + p^2 t - y_0)\right] \qquad E = \frac{2}{p}\tanh\left[\frac{p}{2}(y + p^2 t - y_0)\right] + \frac{2}{p}\,.$$

Now, let us consider what happenes in the multisoliton case. Taking a decreasing solution of the mKdV equation (11) in the form

$$\Theta(y,\tau) = \Xi(\eta_1, \ldots \eta_N) \qquad \eta_k = p_k y - p_k^3 \tau - \zeta_k^0 \quad k = 1, \ldots, N \tag{17}$$

where p_k, ζ_k^0 are real constants, due to (12) one easily finds the function $F(\eta_1, \ldots \eta_N) \equiv R(y,\tau)$. But for the complete description of the HDE solution one needs (15), that is the function $G(\eta_1, \ldots, \eta_N)$:

$$G(\eta_1, \ldots, \eta_N) \boxminus E(y,\tau)\,.$$

As follows from (13), this function is to be calculated by

$$G(\eta_1, \ldots, \eta_N) = \int_{-\infty}^{y} [1 - F(\eta_1, \ldots, \eta_N)] \, \mathrm{d}y \, . \tag{18}$$

For arbitrary N the function $F(\eta_1, \ldots, \eta_N)$ obtained by means of the mKdV solution $\Theta(y, \tau)$ has rather complicated dependence on y and the analytic calculation of G seems to be extremely difficult. Therefore the traditional use of the links between the HDE and mKdV (or KdV) equation, although it gives the parametric representation (18) for G, is not effective for the explicit construction of the HDE multisoliton solutions from an analytic point of view.

When constructing the HDE multisoliton solutions by means of the IST [6], one meets the analytical difficulties of the same level. The solution of the Gel'fand–Levitan equations, contained in the paper mentioned, gives explicitly only the function $F(\eta_1, \ldots, \eta_N)$. It reads as follows:

$$F(\eta_1, \ldots, \eta_N) = \left[1 - \sum_{m=1}^{N} \frac{\det B^{(m)}(\eta_1, \ldots, \eta_N)}{\det B(\eta_1, \ldots, \eta_N)}\right]^2$$

where η_k is given by (17), (14) and (3), B is an $N \times N$ matrix with matrix elements

$$B_{nm} = \delta_{nm} + \frac{2p_n}{p_n + p_m} \exp(-\eta_n)$$

and the matrix $B^{(m)}$ is obtained from B by changing the mth column by $2\exp(-\eta_n)$. The function $G(\eta_1, \ldots, \eta_N)$ which is necessary for the complete description of the N-soliton solution is again to be calculated by (18). Thus the problem is also reduced to handling with the reciprocal transformation and this creates impenetrable analytic difficulties.

The approach proposed in the present paper allows one to obtain a representation for the function $F(\eta_1, \ldots, \eta_N) \equiv R(y, \tau)$, such that an explicit calculation of the reciprocal transformation (18) becomes a trivial analytic problem.

3. The scheme of constructing the HDE soliton solution and final results

The main idea of our approach to constructing the N-soliton solution of the HDE is to assume it to depend on $N - 1$ additional parameteres—'higher times' $t_2, \ldots, t_N$. In other words we integrate the system which consists from the first N higher-order analogues of the HDE [5, 16]

$$r_{t_n} = r^3[-\partial_x^3 r I r]^n r^{-3} r_x \qquad n = 1, \ldots N \tag{19}$$

where the operator I is defined by

$$I\varphi(x, t) = \int_x^{\infty} \varphi(x', t) \, \mathrm{d}x' \, .$$

Now by t we denote the vector $t = (t_1, \ldots, t_N)$. Notice that the dynamics of the HDE itself is described in terms of the component t_1.

We shall seek a 'decreasing' ($r \to 1$ as $|x| \to \infty$) solution of the Nth-order HD system (19) in the form

$$r(x,t) = F(\zeta)$$

where

$$\zeta \equiv (\zeta_1, \ldots, \zeta_N)$$

$$\zeta_k = p_k[x + \varepsilon(x,t)] + \sum_{m=1}^{N} p_k^{2m+1} t_m - \zeta_k^0 \quad k = 1, \ldots, N \tag{20}$$

p_k, ζ_k^0, $k = 1, \ldots, N$, are real constants and the phase function $\varepsilon(x,t)$ related to $r(x,t)$ by (3) is defined implicitly by

$$\varepsilon(x,t) = G(\zeta)\,. \tag{21}$$

Our aim is to find the functions $F(\zeta)$, $G(\zeta)$ explicitly. Let us subsequently differentiate (21) with respect to t_m, $m = 1, \ldots, N$. This yields the set of equations

$$\sum_{k=1}^{N} [p_k^{2m} + \varepsilon_{t_m}(x,t)] p_k G_{\zeta_k}(\zeta) = \varepsilon_{t_m}(x,t) \qquad m = 1, \ldots, N \tag{22}$$

where subscripts t_m, ζ_k denote the corresponding partial derivatives.

These equations can be considered as the Nth-order algebraic system on $G_{\zeta_k}(\zeta)$, $k = 1, \ldots, N$. It is easy to verify that its solution has the form

$$p_k G_{\zeta_k}(\zeta) = \frac{\sum_{m=1}^{N} A_{mk} \varepsilon_{t_m}(x,t)}{1 + \sum_{m,k=1}^{N} A_{mk} \varepsilon_{t_m}(x,t)} \tag{23}$$

where

$$A_{mk} = \Delta_{mk} / \det P\,. \tag{24}$$

Here P is a $N \times N$ matrix with the entries $P_{mk} = p_k^{2m}$, Δ_{mk} is an algebraic adjunct of the element P_{mk}.

Now we differentiate (21) with respect to x and make use of the relation $\varepsilon_x + 1 = 1/r$, which follows from (3). In a result we obtain

$$r(x,t) = 1 - \sum_{k=1}^{N} p_k G_{\zeta_k}(\zeta)\,.$$

Further substitution of (23) to the last equation gives

$$r(x,t) = \frac{1}{1 + \sum_{m=1}^{N} \alpha_m \varepsilon_{t_m}(x,t)} \tag{25}$$

where

$$\alpha_m = \sum_{k=1}^{N} A_{mk}\,. \tag{26}$$

A_{mk} is given by (24). Notice that up to now we have not used the fact that $r(x,t)$ has to obey the HD system (19). The obtained relation (25) is a consequence of the phase function $\varepsilon(x,t)$ definition (3), the form of the functional equation (21) on $\varepsilon(x,t)$ and simple operations with the algebraic system (22).

Now, using the fact that $r(x,t)$ has to solve the system (19), we are going to find some new expressions for the quantities $\varepsilon_{t_m}(x,t)$, $m = 1, \ldots, N$. To do this we shall use the known [16] links between the Dym and the KdV hierarchies.

Lemma 1. Let the function

$$U(y,\tau) = -2\partial_y^2 \ln \Omega(y,\tau) \tag{27}$$

obey the system, which consists of the first N 'higher' KdV analogues

$$U_{\tau_n} = -L^n U_y \qquad n = 1, \ldots, N \tag{28}$$

where the operator L is defined by

$$L\star = \partial_y^2 - 4U + 2U_y \int_y^\infty \star \mathrm{d}y' .$$

It is assumed that $\partial_y \ln \Omega \to 0$ as $|y| \to \infty$.

Let the function $r(x,t)$, $r \to 1$ as $|x| \to \infty$, obey the HD system (19), which is reciprocally associated [16] with the KdV system (28). In particular, the reciprocal links between these systems imply that

$$y = x + \varepsilon(x,t) \qquad \tau = -t \tag{29}$$

where $\varepsilon(x,t)$ is given by (3).

Then there exist the following relations between the quantities related to each of these systems:

$$\varepsilon_{t_1}(x,t) = -4\partial_y^2 \ln \Omega(y,\tau) \equiv 2U(y,\tau) \tag{30}$$

$$\varepsilon_{t_m}(x,t) = 4\partial_{\tau_{m-1}} \partial_y \ln \Omega(y,\tau) \qquad m = 2, \ldots, N \tag{31}$$

where the change of variables (29) is implied.

Proof. As follows from [16] the solution U of the KdV system (28) is linked with the solution $r(x,t)$ of the HD system (19) as follows:

$$\varepsilon_{t_1}(x,t) \boxminus - \int_{-\infty}^{x} \frac{r_{t_1}}{r^2} \mathrm{d}x' = 2U(y,\tau) \tag{32}$$

where the change of variables (29) which links the above system is implied. The last relation is exactly (30). To obtain (31) we make use of the fact [16] that the higher KdV analogues can be written in terms of the quatities ε_{t_m}:

$$U_{\tau_n}(y,\tau) = -\tfrac{1}{2}\partial_y \varepsilon_{t_{n+1}}(x,t) \qquad n = 1, \ldots, N-1 . \tag{33}$$

Now let us insert the function U written in the form (27) to the RSH of the last equation and take into account the boundary conditions $\partial_y \ln \Omega \to 0$ and $\varepsilon_{t_n} \to 0$ as $|y| \to \infty$. The first condition is indicated in the formulation of lemma 1. The second one follows from the fact that $\varepsilon(x,t) \to 0$ as $x \to -\infty$, $\varepsilon(x,t) \to \varepsilon_0 = \int_{-\infty}^{\infty} (r^{-1} - 1)\,\mathrm{d}x < \infty$, hence $\varepsilon_{t_m}(x,t) \to 0$ as $|x| \to \infty$, and due to (29) $|y| \to \infty$ as $|x| \to \infty$. These conditions allow one to take off the derivative with respect to y in both sides of (33) and thereby to obtain (31). Now let us again return to the relation (25) which upon use of (30) and (31) and the change of variables (29) can be written as

$$r(x,t) \equiv R(y,\tau) = \frac{1}{1 + 4\sum_{m=1}^{N} \alpha_m \partial_{\tau_{m-1}} \partial_y \ln \Omega(y,\tau)} \tag{34}$$

where $\partial_{\tau_0} \boxminus -\partial_y$.

The next step is to choose an appropriate solution $U(y,\tau)$ (equation (27)) of the KdV system (28). This choice is controlled by the following.

Lemma 2. Let a solution $U(y,\tau)$ of the KdV system (28) in the vicinity of points $y_k \equiv y_k(\tau)$, $k = 1,\dots,N$, have the asymptotic behaviour

$$U(y,\tau) = \frac{2}{(y-y_k)^2} + \mathrm{O}(1) \qquad k = 1,\dots,N. \tag{35}$$

Then the solution $r(x,t)$ of the HD system (19) which corresponds to it under the transformation (29) in the vicinity of the points $x_k(t)$, $k = 1,\dots N$, defined by $y_k = x_k + \varepsilon(x_k, t)$, is characterized by the asymptotic behaviour

$$r(x,t) = C_k(x-x_k)^{2/3} + \mathrm{O}((x-x_k)^{4/3}) \tag{36}$$

where C_k are some constants.

Proof. Since $r_{t_1} = r^3 r_{xxx}$, $y = x + \varepsilon(x,t)$ and $\partial_y = (1/R)\partial_x$, where $R(y,\tau) \equiv r(x,t)$, $\tau = -t$, (32) can be written as

$$U = \frac{3}{4}\left(\frac{R_y}{R}\right)^2 - \frac{1}{2}\frac{R_{yy}}{R}. \tag{37}$$

Now by the simple direct calculation one can show that for the function $U(y,\tau)$ to have the expansion (35) the function $R(y,\tau)$ in (37) has to be characterized as $y \sim y_k$ by the asymptotic behaviour

$$R(y,\tau) = a_k(y-y_k)^2 + \mathrm{O}((y-y_k)^4) \qquad k = 1,\dots,N \tag{38}$$

where a_k are arbitrary constants.

To rewrite the last relation in terms of the variable x let us notice on the basis of (29) that

$$y - y_k = x - x_k + E(y,\tau) - E(y_k,\tau) \tag{39}$$

where $E(y,\tau) \equiv \varepsilon(x,t)$. Since $E_y = 1 - R$, one can easily find the expansion for $E(y,\tau)$ and on use of (39) one obtains

$$y - y_k = \left(\frac{3}{a_k}\right)(x-x_k)^{1/3} + \mathrm{O}((x-x_k)^{2/3}). \tag{40}$$

Finally, the insertion of the last relation to (38) leads to the representation (36) with $C_k = (9a_k)^{1/3}$.

Notice that in the one-soliton case $R(y,\tau) = \tanh^2\left[\frac{1}{2}p(y - y_1(\tau))\right]$ (see (1)). Thus, in this case $a_1 = \left(\frac{1}{2}p\right)^2$ and the cusp soliton in the vicinity of its zero has the form

$$r \sim \left(\tfrac{3}{2}p\right)^{2/3}(x - x_1(t))^{2/3} \qquad x \to x_1(t).$$

The corresponding KdV solution $U(y,\tau)$ related to R by (37) reads

$$U(y,\tau) = \frac{p^2}{2\sinh^2\left[\frac{1}{2}p(y-y_1(\tau))\right]}.$$

In the vicinity of y_1 it is characterized by the asymptotic behaviour (35) and is known to be linked with the KdV one-soliton solution $\hat{U}(y,\tau) = -\frac{1}{2}p^2\cosh^{-2}\left[\frac{1}{2}p(y-y_1(\tau))\right]$ by the auto-Bäcklund transformation [5]

$$U+\hat{U} = \tfrac{1}{2}\left[\int_{-\infty}^{y}(U-\hat{U})\,\mathrm{d}y'\right]^2 .$$

For arbitrary N the situation is similar to that in the case $N = 1$. The solution of the KdV system which has N double poles $y_k(\tau)$ (equation (35)) is linked with the N-soliton solution by the same auto-Bäklund transformation. This solution can be written in the form (27) with

$$\Omega(y,\tau) = \sum_{m\in Z^N}{}' \exp\left(\tfrac{1}{2}\langle Bm, m\rangle + \langle \eta + \mathrm{i}\pi m, m\rangle\right) \tag{41}$$

where the components η_k of the vector $\eta = (\eta_1, \ldots, \eta_N)$ are given by (20) with the change $x + \varepsilon(x,t) = y,\ t = -\tau$. Here B is an $N \times N$ symmetric matrix with the entries

$$B_{kl} = 2\ln\left|\frac{p_k - p_l}{p_k + p_l}\right| \qquad B_{kk} = 0 .$$

By $\sum'$ we denote the summation over the vectors $m \in Z^N$ which components are equal to 0 or 1, $\langle\cdot,\cdot\rangle$ is the standard inner product: $\langle \eta, m\rangle = \sum_{k=1}^{N}\eta_k m_k$.

Now lemmas 1 and 2 guarantee that the function $\Omega(y,\tau)$ given by (41) generates by means of (34) the function $R(y,\tau)$ which has double zeros at the points $y_k(\tau),\ k = 1,\ldots,N$ (see (38)) and due to (40) produces the solution $r(x,t)$ of the KdV system with N 'cusps' (36) at the corresponding points $x_k(t)$.

But (34) with $\Omega(y,\tau)$ given by (41) gives only a rather difficult integro-functional equation on this N-cusp solution since

$$y = x + \varepsilon(x,t) \equiv x + \int_{-\infty}^{x}\left(\frac{1}{r(x',t)} - 1\right)\mathrm{d}x' .$$

It can be written in the form (14)–(16) with $R(y,\tau)$ explicitly given by (34) (now it is assumed that $t = (t_1,\ldots,t_N),\ \tau = (\tau_1,\ldots,\tau_N)$). But the representation obtained for $R(y,\tau)$ is not appropriate for analytical handling with the reciprocal transformation (13). We remind ourselves that it is this point which was crucial in using other approaches to the explicit construction of the HDE multisoliton solutions discussed in section 2 (that is the traditional use of the links between the HDE and the KdV or mKdV equations and the IST application). The approach proposed in the present paper has the advantage that the mentioned problem can be easily solved. This will be done by passing from (34) to such representation for $R(y,\tau)$ which makes the calculation of the reciprocal transformation trivial.

Theorem. The function $R(y,\tau)$ given by (34) can be represented in the form

$$R(y,\tau) = 1 + 4\sum_{m=1}^{N}\alpha_m\partial_{\tau_{m-1}}\partial_y \ln f(y,\tau) \tag{42}$$

where the function $f(y,\tau)$ is determined by the N-soliton soliton of the KdV system:

$$\hat{U}(y,\tau) = -2\partial_y^2 \ln f(y,\tau)$$

and reads as follows [24]:

$$f(y,\tau) = \sum\nolimits'_{m\in Z^N} \exp\left(\tfrac{1}{2}\langle Bm, m\rangle + \langle \eta, m\rangle\right) . \tag{43}$$

Here all the designations are the same as in (41).

Proof. The solution $U(y,\tau)$ and $\hat{U}(y,\tau)$ of the KdV system linked by the auto-Bäcklund transformation are known [16] to generate via reciprocal links the solutions of the HD system $r(x,t)$ and $r'(x',t')$ respectively, the latter being linked by the reciprocal auto-Bäcklund transformation for the HD hierarchy. In particular, this implies that

$$r^{-1}(x,t) = r'(x',t') \tag{44}$$

and $y \equiv x + \varepsilon(x,t) = x' + \varepsilon'(x',t')$, $\tau = -t' = -t$, where ε' is related to r' similalyr to ε and r.

Applying now all the constuctions which led to (34) to the pair $\hat{U}(y,\tau) \leftrightarrow r'(x',t')$ one obtains

$$r'(x',t') \equiv R'(y,\tau) = \frac{1}{1 + 4\sum_{m=1}^{N} \alpha_m \partial_{\tau_{m-1}} \partial_y \ln f(y,\tau)}$$

where $f(y,\tau)$ is given by (43). Finally, the link $R'(y,\tau) = 1/R(y,\tau)$ which follows from (44) leads to (42).

On the basis of such representation for $R(y,\tau)$ the calculation of the reciprocal transformation (13) is trivial. Taking into account that $\partial_{\tau_{m-1}} \ln f(y,\tau)|_{y=-\infty} = 0$ one easily obtains

$$E(y,\tau) = -4\sum_{m=1}^{N} \alpha_m \partial_{\tau_{m-1}} \ln f(y,\tau) \qquad \partial_{\tau_0} \equiv -\partial_y .$$

Since

$$\eta_k = p_k y - \sum_{m=1}^{N} p_k^{2m+1} t_m - \eta_k^0 \qquad k = 1,\ldots,N \tag{45}$$

one has that $\partial_y = \sum_{k=1}^{N} p_k \partial_{\eta_k}$, $\partial_{\tau_{m-1}} = -\sum_{k=1}^{N} p_k^{2m-1} \partial_{\eta_k}$, $m = 2,\ldots,N$. Thus the equation for $E(y,\tau)$ can be written as

$$E(y,\tau) = \frac{4\sum_{k=1}^{N} \beta_k \partial_{\eta_k} f(y,\tau)}{f(y,\tau)} \tag{46}$$

where

$$\beta_k = \sum_{m=1}^{N} \alpha_m p_k^{2m-1} . \tag{47}$$

Using the definition (26), (24) of the constants α_m on can show (see appendix) that

$$\beta_k = \frac{1}{p_k}\,. \tag{48}$$

Finally, the equation (46) defining implicitly the phase function $\varepsilon(x,t)$, on use of (43), can be written as

$$\varepsilon(x,t) \equiv E(y,\tau) == 4\frac{\sum'_{m\in Z^N} \exp(\frac{1}{2}\langle Bm,m\rangle + \ln\langle\beta,m\rangle + \langle\eta,m\rangle)}{\sum'_{m\in Z^N} \exp(\frac{1}{2}\langle Bm,m\rangle + \langle\eta,m\rangle)} \tag{49}$$

where $\beta \in \mathbf{R}^N$ has the components (48), all other designations are the same as in (41).

Knowing the solution $\varepsilon(x,t)$ of the functional equation (49), the solution $r(x,t)$ of the HD system is given explicitly by the equation

$$r(x,t) \equiv R(y,\tau) = \left[\frac{\sum'_{m\in Z^N} \exp(\frac{1}{2}\langle Bm,m\rangle + \langle\eta + \mathrm{i}\pi m,m\rangle)}{\sum'_{m\in Z^N} \exp(\frac{1}{2}\langle Bm,m\rangle + \langle\eta,m\rangle)}\right]^2 . \tag{50}$$

Now we show how this representation for $R(y,\tau)$ follows from that obtained earlier, i.e. (34) and (42). To this end we rewrite (42) in the form

$$R(y,\tau) = \frac{W[f]}{f^2(y,\tau)} \tag{51}$$

and (34) in the form

$$R(y,\tau) = \frac{\Omega^2(y,\tau)}{W[\Omega]} \tag{52}$$

where

$$W[f] = f^2 - 4\sum_{k=1}^{N} p_k^{-1}[f(\partial_y\partial_{\eta_k} f) - (\partial_y f)(\partial_{\eta_k} f)]\,.$$

$W[\Omega]$ is the similar expression where f is substituted by Ω.

It follows from the explicit formulae for f and Ω, that in both fractions (51) and (52) the numerators and denominators are linear combinations of $\exp(\sum_{k=1}^{N} n_k\eta_k)$, where each of the integers n_k may take the value of 0, 1 or 2. Moreover, these fractions are equal iff

$$W[f] = \Omega^2 \cdot C \tag{53}$$

and

$$W[\Omega] = f^2 \cdot \frac{1}{C} \tag{54}$$

where C is also a linear combination of the same exponents.

Further, let us introduce the formal operation J:

$$J\exp\left(\sum_{k=1}^{N} n_k\eta_k\right) = \prod_{k=1}^{N}(-1)^{n_k}\exp\left(\sum_{k=1}^{N} n_k\eta_k\right).$$

Comparing (43) for f with (41) for Ω one sees that $f = J\Omega$. Further use of the above remark on the structure of the fractions (51) and (52) leads to the relations

$$f^2 = J\Omega^2 \qquad W[\Omega] = JW[f]. \tag{55}$$

Now let us apply the operation J to the relation (53). Due to (55) this gives

$$W[\Omega] = f^2 \cdot JC.$$

Comparison of the last relation with (54) shows that

$$\frac{1}{C} = JC.$$

By virtue of C being a linear combination of $\exp(\sum_{k=1}^{N} n_k\eta_k)$, where n_k may be equal to 0, 1 or 2, the last relation holds only if $C \equiv 1$. Now, on use of (53) and (51) we obtain the relation

$$R(y,\tau) = \left[\frac{\Omega(y,\tau)}{f(y,\tau)}\right]^2$$

and thereby (50).

Equations (49) and (50) describe the N-cusp soliton solution of the HD system (19) completely. The phase function $\varepsilon(x,t)$ is defined implicitly by (49), where the components of the vector η are given by (45), $y = x + \varepsilon(x,t)$. The HD system solution $r(x,t)$ itself is given explicitly by (50).

Remark 1. It is clear that if $t_m = 0$, $m = 2,\ldots,N$, (49)–(50) give the N-cusp soliton solution of the HDE itself. In what follows in this section we imply this case ($t_m = 0, m \geqslant 2$).

Remark 2. It can be easily seen that the RSH of (50) tends to 1 as $|x| \to \infty$. The RSH of (49) tends to $4\sum_{k=1}^{N} p_k^{-1}$ as $x \to +\infty$. Since the function $\varepsilon(x,t)$ is defined by (3) this gives the value of the integral

$$\int_{-\infty}^{\infty}\left(\frac{1}{r(x',t)} - 1\right)\mathrm{d}x' = 4\sum_{k=1}^{N}\frac{1}{p_k}. \tag{56}$$

In the one-soliton case (see (1) and (2)) this conserving quantity is simply the mass of the soliton

$$\int_{-\infty}^{\infty}\left(\frac{1}{r(x',t)} - 1\right)\mathrm{d}x' = \frac{4}{p_1}.$$

The formula (56) implies that the mass of N colliding HDE solitons is equal to the sum of the masses of each of them.

Now we shall consider a simple sample. Equations (49) and (50) at $N = 1$ yield

$$r(x,t) = \left(\frac{1-\exp\zeta_1}{1+\exp\zeta_1}\right)^2 = \tanh^2\left(\frac{\zeta_1}{2}\right)$$

$$\varepsilon(x,t) = \frac{4}{p_1}\frac{\exp\zeta_1}{1+\exp\zeta_1} = \frac{2}{p_1}\tanh\left(\frac{\zeta_1}{2}\right) + \frac{2}{p_1}$$

where $\zeta_1 = p_1[x+\varepsilon(x,t)] + p_1^3 t - \zeta_1^0$. Thus, we obtained the well known cusp soliton of the HDE [6].

For $N = 2$ one has

$$r(x,t) = \left[\frac{1-\exp\zeta_1 - \exp\zeta_2 + \gamma\exp(\zeta_1+\zeta_2)}{1+\exp\zeta_1+\exp\zeta_2+\gamma\exp(\zeta_1+\zeta_2)}\right]^2 \tag{57}$$

$$\varepsilon(x,t) = 4\frac{p_1^{-1}\exp\zeta_1 + p_2^{-1}\exp\zeta_2 + (p_1^{-1}+p_2^{-1})\gamma\exp(\zeta_1+\zeta_2)}{1+\exp\zeta_1+\exp\zeta_2+\gamma\exp(\zeta_1+\zeta_2)} \tag{58}$$

where $\zeta_k = p_k[x+\varepsilon(x,t)] + p_k^3 t - \zeta_k^0,\ k = 1,2$ and

$$\gamma = \left(\frac{p_2-p_1}{p_2+p_1}\right)^2 .$$

It is easy to show that solution (57), (58) splits into two cusp solitons as $t \to^{\pm} \infty$. The appropriate asymptotics look like (we assume $p_2 > p_1$)

$$r(x,t) \sim_{t\to\pm\infty} r^{\pm}(x,t) = \prod_{k=1}^{2}\tanh^2\left(\tfrac{1}{2}\zeta_k^{\pm} - \delta_k^{\pm}\right)$$

where

$$\zeta_k^{\pm} = p_k[x+\varepsilon_{\rm as}^{\pm}(x,t)] + p_k^3 t - \zeta_k^0 \qquad k = 1,2$$

and the asymptotic phase function $\varepsilon_{\rm as}^{\pm}(x,t)$ is defined implicitly by the equation

$$\varepsilon_{\rm as}^{\pm}(x,t) = \sum_{k=1}^{N}\left[\frac{2}{p_k}\tanh\left(\frac{1}{2}\zeta_k^{\pm} - \delta_k^{\pm}\right) + \frac{2}{p_k}\right].$$

Here

$$\delta_1^- = -\tfrac{1}{2}\ln\gamma \qquad \delta_2^- = 0 \qquad \delta_1^+ = 0 \qquad \delta_2^+ = -\tfrac{1}{2}\ln\gamma .$$

4. Concluding remarks

Our method can be called the 'higher-times approach' (HTA) since for the effective analytical construction of the N-soliton solution of the HDE the higher HDE analogues were used essentially. As it follows from (19) the dynamics of each equation from the Dym hierarchy is described in terms of its own time t_m, $m = 1, \ldots, N$. Such approach to the HDE integration gives the necessary relations for the 'effectivization' of the links between the KdV (or mKdV) and the HDE.

An important advantage of the HTA is invariance of its main steps with respect to boundary conditions for the HD hierarchy. In particular, in the framework of this approach the finite-gap solutions of the HDE and its higher analogues can be constructed explicitly. This will be done in a seperate paper.

In conclusion we make some remarks. The solution $r'(x', t')$, which is linked with the N-cusp soliton solution by a reciprocal auto-Bäcklund transformation introduced for the HDE in [16], turns out to be a multi-valued function on x' with 2^N branches. These branches can be connected in N cross-points and yield a continuous bounded function $\hat{r}'(x', t')$: $1 \leqslant \hat{r}' \leqslant A$, where A is a positive solution of the equation $A = \coth A$. But the discussion of this solution is out the scope of the present paper.

We would like also to mark the known results [18, 23, 25] concerning the solutions of the HDE with other boundary conditions. The simplest periodic solutions were obtained explicitly in [23, 25]. In [18] one can find the method which allows starting from the N-soliton solution of the KdV equation to construct on finite intervals $[0, a_i]$, $i = 1, \ldots, N$ (the constants a_i are defined by the asymptotic speeds of the KdV N-soliton solution) N special solutions of the HDE, these solutions being obtained in parametric form by reciprocal transformation from N 'interecting' solitons of the KdV equation.

Acknowledgments

The author is indebted to Professor Yu A Kuperin for discussions, especially for drawing the author's attention to this problem. I also would like to thank Professor M S Birman for interesting remarks. This work is supported by Russian Fundamental Foundation under Grant No 93-011-266.

Appendix

In this appendix we are going to prove that

$$\beta_k = 1/p_k$$

where β_k is defined by (47). We remind ourselves that due to (26) and (24), $\alpha_m = \sum_{k=1}^{N} \Delta_{mk}/\det P$, where P is an $N \times N$ matrix with the entries $P_{mk} = p_k^{2m}$ (m is the line number, k is the column number), Δ_{mk} is an algebraic adjunct of the element P_{mk}.

From the definition of the matrix P one sees that $\det P = \prod_{k=1}^{N} p_k^2 W(p_1^2, p_2^2, \ldots, p_N^2)$, where W is the Vandermonde determinant. Hence

$$\det P = \prod_{k=1}^{N} p_k^2 \prod_{i>j} (p_i^2 - p_j^2) . \tag{A1}$$

Further, $\sum_{k=1}^{N}\Delta_{mk} = \det P^{(m)}$, where the matrix $P^{(m)}$ is obtained from P by changing the mth line by units. One can show that

$$\det P^{(m)} = (-1)^{m-1} \sum_{{}_N C_{N-m}} p_{i_1}^2 \cdots p_{i_{N-m}}^2 \prod_{i>j}(p_i^2 - p_j^2) \quad \text{(A2)}$$

where ${}_N C_{N-m}$ indicates summation over all possible combination of $N-m$ integers taken from the first N integers. To prove (A2) first of all we notice that

$$\det P^{(m)} = (-1)^{m-1} \det P'_m \quad \text{(A3)}$$

where P'_m is the $N \times N$ matrix in which the first m lines are given as $(p_1^{2k-2}, p_2^{2k-2}, \dots, p_N^{2k-2})$, $k = 1, \dots m$, and the last $N-m$ lines have the form $(p_1^{2k}, p_2^{2k}, \dots, p_N^{2k})$, $k = m+1, \dots, N$. Futher, we consider the $(N+1)\times(N+1)$ Vandermonde determinant

$$W(z, p_1^2, p_2^2, \dots, p_N^2) = \prod_{i>j}(p_i^2 - p_j^2)\prod_s (p_s^2 - z) \equiv \sum_{k=1}^{N} z^k A_k . \quad \text{(A4)}$$

It is easy to see that from one side $A_m = (-1)^m \det P'_m$ and due to (A4)

$$A_m = (-1)^m \sum_{{}_N C_{N-m}} p_{i_1}^2 \cdots p_{i_{N-m}}^2 .$$

Hence on use of (A3) one obtains (A2). Inserting this equation into the definition of α_m and using (A1) we see that

$$\alpha_m = (-1)^{m-1} \sum_{{}_N C_{N-m}} p_{i_1}^2 \cdots p_{i_{N-m}}^2 \Big/ \prod_{k=1}^{N} p_k^2 .$$

Now we remenber that the constants β_k are given by (47). Hence

$$\beta_k = \frac{\mu}{p_k} \Big/ \prod_{s=1}^{N} p_s^2$$

where

$$\mu = \sum_{m=1}^{N} (-1)^{m-1} \sum_{{}_N C_{N-m}} p_{i_1}^2 \cdots p_{i_{N-m}}^2 p_k^{2m} . \quad \text{(A5)}$$

Thus, to prove the relation $\beta_k = 1/p_k$ one needs to show that

$$\mu = \prod_{s=1}^{N} p_s^2 . \quad \text{(A6)}$$

This can be done by means of the following calculation:

$$\text{RSH of (A5)} = \sum_{m=1}^{N-1} (-1)^{m-1} p_k^{2(m+1)} \sum_{\substack{i_s \neq k \\ {}_N C_{N-m}}} p_{i_1}^2 \cdots p_{i_{N-m-1}}^2$$
$$+ \sum_{m=2}^{N} (-1)^{m-1} p_k^{2m} \sum_{\substack{i_s \neq k \\ {}_N C_{N_m}}} p_{i_1}^2 \cdots p_{i_{N-m}}^2 + \prod_{s=1}^{N} p_s^2 .$$

Changing now in the first sum the summation index m by $m-1$ one obtains the second sum with the opposite sign. Thus, (A6) and thereby $\beta_k = p_k^{-1}$ are proved.

References

[1] Kruskal M D 1975 *Lecture Notes in Physics* ed J Moser (Berlin: Springer)
[2] Wadati M, Konno K and Ichikawa Y H 1979 *J. Phys. Soc. Japan* **47** 1698
[3] Sabatier P C 1979 *Lett. Nuovo Cimento* **26** 483
[4] Yi-Shen L 1982 *Lett. Nuovo Cimento* **70** 1
[5] Calogero F and Degasperis A 1982 *Spectral Transform and Solitons I, Studies in Mathematics and Applications* (Amsterdam: North-Holland)
[6] Wadati M , Ichikawa Y H and Shimizu 1980 *Prog. Theor. Phys.* **64** 1959
[7] Magri F 1978 *J. Math. Phys.* **19** 1156
[8] Ibragimov N H 1985 *Transformation Groups Applied to Mathematical Physics* (Dordrecht: Reidel)
[9] Leo M , Leo R A, Soliam G and Solombrino L 1983 *Phys. Rev.* D **27** 1406
[10] Leo M, Leo R A, Soliani G and Solombrino L 1983 *Lett. Nuovo Cimento* **38** 45
[11] Kawamoto S 1985 *J. Phys. Soc. Japan* **54** 2055
[12] Ishimori J 1982 *J. Phys. Soc. Japan* **51** 3036
[13] Ibragimov N H 1981 *C. R. Acad. Sci. Paris* **293** 657
[14] Weiss J 1983 *J. Math. Phys.* **24** 1405
[15] Weiss J 1986 *J. Math. Phys.* **27** 1293
[16] Rogers C and Nucci M C 1986 *Phys. Scr.* **33** 289
[17] Guo Ben-Yu and Rogers C 1989 *Sci. China* **32** 283
[18] Fuchssteiner B, Schulze T and Carillo S 1992 *J. Phys. A: Math. Gen.* **25** 223
[19] Hereman W, Banerjee P P and Chattergee M R 1989 *J. Phys. A: Math. Gen.* **22** 241
[20] Fujimoto A and Watanabe Y 1989 *Phys. Lett.* **136A** 294
[21] Constantin G L and Kadanoff L P 1990 *Physica* **47D** 50
[22] Kadanoff L P 1990 *Phys. Rev. Lett.* **65** 2986
[23] Vasconcelos G L and Kadanoff L P 1991 Note on the Harry Dym equation *Preprint* University of Chicago
[24] Ablowitz M J and Segur H 1981 *Solitons and the Inverse Scattering Transform* (Philadelphia, PA: SIAM)
[25] Chowdhury A R and Mukherjee R 1984 *Phys. Scr.* **29** 293